

ON THE ESCAPE RATE OF UNIQUE BETA-EXPANSIONS

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ABSTRACT. Let $1 < \beta \leq 2$. It is well-known that the set of points in $[0, 1/(\beta - 1)]$ having unique β -expansion, in other words, those points whose orbits under greedy β -transformation escape a hole depending on β , is of zero Lebesgue measure. The corresponding escape rate is investigated in this paper. A formula which links the Hausdorff dimension of univoque set and escape rate is established in this study. Then we also proved that such rate forms a devil's staircase function with respect to β .

Key Words: escape rate, beta-expansion, open system

AMS Subject Classification: 37E05, 11A63

1. INTRODUCTION

1.1. History and motivation: beta shifts. There are many ways to represent real numbers such as decimal expansion, binary expansion etc. In 1957, Rényi [21] generalized the expansions with integer bases to any base $\beta > 1$ including non-integer bases.

Let $1 < \beta \leq 2$ and $x \geq 0$. Write

$$(1) \quad x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{\beta^n} \quad \text{with } \varepsilon_n \in \{0, 1\} \text{ for all } n \geq 1.$$

We call this expression or the sequence $(\varepsilon_1, \dots, \varepsilon_n, \dots)$ a β -*expansion* of x . It is obvious that the smallest number which can be represented by 1 is 0 and the largest one is $\frac{1}{\beta-1}$.

Let $J_\beta = [0, \frac{1}{\beta-1}]$ and

$$\Omega_\beta = \{0, 1\}^{\mathbb{N}} := \{(w_1, \dots, w_n, \dots) : w_n = 0 \text{ or } 1 \text{ for all } n \geq 1\}.$$

Denote by $\Omega_\beta^n = \{0, 1\}^n$ and $\Omega_\beta^* = \cup_{n=1}^{\infty} \Omega_\beta^n$. Let \prec and \preceq be the *lexicographical order* on Ω_β . More precisely, $w \prec w'$ means that there exists $k \in \mathbb{N}$ such that $w_i = w'_i$ for all $1 \leq i < k$ and $w_k < w'_k$, meanwhile, $w \preceq w'$ means that $w \prec w'$ or $w = w'$. This order can

be extended to Ω_β^* by identifying a finite block (w_1, \dots, w_n) with the sequence $(w_1, \dots, w_n, 0^\infty)$. The topic on the number of β -expansions of reals in J_β is very popular since Erdős investigated the reals with unique β -expansion in 1990s.

Theorem 1.1 ([11]). *If $1 < \beta < \frac{1+\sqrt{5}}{2}$, then every $x \in (0, \frac{1}{\beta-1})$ has a continuum of different β -expansions.*

Theorem 1.2 ([22]). *If $\frac{1+\sqrt{5}}{2} \leq \beta < 2$, then λ -almost every $x \in (0, \frac{1}{\beta-1})$ has a continuum of different β -expansion, where λ is the Lebesgue measure on \mathbb{R} .*

Define a projection map $\pi_\beta : \Omega_\beta \rightarrow J_\beta$ as

$$\pi_\beta(w) = \sum_{n=1}^{\infty} \frac{w_n}{\beta^n}$$

for $w = (w_1, \dots, w_n, \dots) \in \Omega_\beta$. Then $\#\pi_\beta^{-1}(x)$ is the number of β -expansions of $x \in J_\beta$, here $\#$ denotes the cardinality of a finite set. Denote

$$\mathcal{U}_\beta = \{x \in J_\beta : \#\pi_\beta^{-1}(x) = 1\},$$

that is, the set of the points with unique β -expansion. The set \mathcal{U}_β is called the *univoque set*. Together with Theorem 1.1 and Theorem 1.2, we know that $\lambda(\mathcal{U}_\beta) = 0$ for any $1 < \beta \leq 2$. Glendinning and Sidorov [13] showed a finer description on \mathcal{U}_β as the following.

Theorem 1.3 ([13]). *The set \mathcal{U}_β is*

- *empty if $\beta \in (1, \frac{1+\sqrt{5}}{2}]$;*
- *countable for $\beta \in (\frac{1+\sqrt{5}}{2}, \beta_*)$, where $\beta_* = 1.787231650\dots$ is the Komornik-Loreti constant (see also [16]);*
- *an uncountable Cantor set of zero Hausdorff dimension if $\beta = \beta_*$; and*
- *a Cantor set of positive Hausdorff dimension for $\beta \in (\beta_*, 2)$.*

Recently, Kong and Li [15] gave much more information for \mathcal{U}_β for $\beta_* < \beta < 2$. The authors showed that the Hausdorff dimension of such set forms a devil's staircase function, i.e. $D(\beta) := \dim_H \mathcal{U}_\beta$ is continuous, monotonic and $D' < 0$ almost everywhere. They also gave explicit formula for the Hausdorff dimension of \mathcal{U}_β when β is in any

admissible interval $[\beta_L, \beta_U]$ (see Theorem 2.6 [18] for the definition of such interval).

Among all the β -expansions of any given number $x \in J_\beta$, the maximum and minimum in the sense of lexicographical order are provided by greedy and lazy algorithms respectively. These two algorithms can be induced by greedy and lazy β -transformations respectively.

Definition 1.4 (Greedy β -transformation). Let $1 < \beta \leq 2$. The *greedy β -transformation* $G_\beta : J_\beta \rightarrow J_\beta$ is defined as

$$G_\beta(x) = \begin{cases} \beta x, & \text{if } 0 \leq x < \frac{1}{\beta}; \\ \beta x - 1, & \text{if } \frac{1}{\beta} \leq x \leq \frac{1}{\beta-1}. \end{cases}$$

The system (J_β, G_β) is called the *greedy β -transformation dynamical system*.

A *coding* of any $x \in J_\beta$ according to G_β which includes two branches can be given as follows. Define

$$\varepsilon_1(x, \beta) = \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{\beta}; \\ 1, & \text{if } \frac{1}{\beta} \leq x \leq \frac{1}{\beta-1} \end{cases}$$

for $x \in J_\beta$. That is, the first branch $[0, \frac{1}{\beta})$ of G_β is labelled by 0 and the other one $[\frac{1}{\beta}, \frac{1}{\beta-1})$ is labelled by 1. Denote $\varepsilon_n(x, \beta) := \varepsilon_1(G_\beta^{n-1}x, \beta)$. Then,

$$(2) \quad x = \sum_{n=1}^{\infty} \varepsilon_n(x, \beta) \beta^{-n},$$

which is called the *greedy β -expansion* of $x \in J_\beta$ and denote $\varepsilon(x, \beta) = (\varepsilon_1(x, \beta), \dots, \varepsilon_n(x, \beta), \dots)$. The trapping region of (J_β, G_β) is (I, T_β) , where $I = [0, 1)$ and $T_\beta = G_\beta|_I$. More precisely, $T_\beta I = I$ and for any $1 \leq x \leq \frac{1}{\beta-1}$, there is some $n \in \mathbb{N}$ such that $T_\beta^n(x) \in I$. If $1 \leq x \leq \frac{1}{\beta-1}$, then $\varepsilon(x, \beta) = (1^k, \varepsilon(y, \beta))$ for some $k \geq 1$ and $y = G_\beta^k(x) \in I$.

Definition 1.5 (Lazy β -transformation). The *lazy β -transformation* $L_\beta : J_\beta \rightarrow J_\beta$ is defined as

$$L_\beta(x) = \begin{cases} \beta x, & \text{if } 0 \leq x \leq \frac{1}{\beta(\beta-1)}; \\ \beta x - 1, & \text{if } \frac{1}{\beta(\beta-1)} < x \leq \frac{1}{\beta-1}. \end{cases}$$

Besides the greedy and lazy β -expansions, other β -expansions are called *intermediate β -expansion*, which is rich from Theorem 1.1 and Theorem 1.2. The corresponding β -transformations are also studied recently, for example, see [18].

By the greedy and lazy β -transformations, the whole interval J_β is partitioned to three parts:

$$\begin{aligned} I_0 &= [0, 1/\beta), \Delta_\beta = [1/\beta, 1/\beta(\beta - 1)], \\ I_1 &= (1/\beta(\beta - 1), 1/(\beta - 1)]. \end{aligned}$$

If $x \in I_0$, the first digit of both greedy and lazy expansions of x are 0; since the greedy and lazy expansions are the maximum and minimum among all β -expansions, we know that the first digit of any β -expansion of x is 0. Similarly, if $x \in I_1$, then the first digit of all β -expansions of x are 1. For $x \in \Delta_\beta$, since the first digit of greedy expansion is 1 and the lazy is 0, the first digits of β -expansions of x have two possibilities.

Proposition 1.6. *The univoque set*

$$\begin{aligned} \mathcal{U}_\beta &= \{x \in J_\beta : G_\beta^n(x) \notin \Delta_\beta \ \forall n \geq 0\} \\ &= \{x \in J_\beta : L_\beta^n(x) \notin \Delta_\beta \ \forall n \geq 0\}. \end{aligned}$$

This proposition tell us that the set \mathcal{U}_β consists of the points whose orbit under G_β or L_β will never fall in the hole Δ_β . So \mathcal{U}_β is regarded as a problem of *open system* (*dynamical system with hole and exclusion systems*) and we just need to focus on the greedy β -transformation G_β .

1.2. History and motivation: open systems and escape rate.

Proposition 1.6 reveals that the study of the topological properties of \mathcal{U}_β is equivalent to the study of *open systems*. Such systems have been studied extensively by physicists and mathematicians in many aspects. Main questions in open systems are: how do typical points of the phase space escape from a given hole, what is the speed of escape, and which hole is leaking the most (see [14, 1, 8, 9, 10])? Escape rate is introduced to measure such quantity and we define such rate in our setting. Denote

$$\tilde{\Gamma}_{\beta,n} = \tilde{\Gamma}_n(\Delta_\beta) := \{x \in J_\beta : G_\beta^k(x) \in \Delta_\beta, 0 \leq k \leq n\}.$$

Since the trapping region of (J_β, G_β) is (I, T_β) , we can define

$$\Gamma_{\beta,n} = \Gamma_n(\Delta_\beta) := \{x \in I : T_\beta^k(x) \in \Delta_\beta \cap I \text{ for some } k, 0 \leq k \leq n\}.$$

It is clear that $\mathcal{U}_\beta \cap I = \cap_{n=1}^\infty I \setminus \Gamma_n(\Delta_\beta)$. Since $\lambda(\mathcal{U}_\beta) = 0$, we have $\lim_{n \rightarrow \infty} \lambda(I \setminus \Gamma_n(\Delta_\beta)) = 0$. (Recall that λ is the Lebesgue measure on \mathbb{R} .) Note that $\Gamma_{n+1}(\Delta_\beta) \supset \Gamma_n(\Delta_\beta)$ for all $n \geq 0$, the following limit exists and we can define the *escape rate* as follows

$$e_\beta = \lim_{n \rightarrow \infty} \frac{-\log \lambda(\Gamma_{\beta,n})}{n} \text{ and } E_\beta = \lim_{n \rightarrow \infty} \frac{-\log \lambda(\Gamma_{\beta,n})}{n \log \beta} = \frac{e_\beta}{\log \beta}.$$

Here we also call E_β the *escape rate* if it causes no confusion. The corresponding escape rate \tilde{e}_β and \tilde{E}_β are defined similarly for $\tilde{\Gamma}_n(\Delta_\beta)$.

The aim of this paper is to calculate the rate e_β and describe how e_β changes as β varies. It is worth pointing out that such a problem has been raised by Bundfuss, Kruger and Troubetzkoy (p.23, [7]).

One would like to develop a relationship between the escape rate properties and topological and/or metric invariants of the invariant set.

We emphasize that the central problem of open systems is how to estimate the escape rate and how the escape rate varies when the hole is shrinking to zero. Our problem is a little bit different since the map T_β also changes with respect to β in $(1, 2]$. The following is the main result of this investigation.

Theorem 1.7. *Let $1 < \beta \leq 2$. Then $\dim_H \mathcal{U}_\beta + E_\beta = 1$.*

Corollary 1.8. *Let $1 < \beta \leq 2$.*

- (1) *If $1 < \beta \leq \beta_*$, then $E_\beta = 1$ and $e_\beta = \log \beta$.*
- (2) *If $\beta_* < \beta \leq 2$, then E_β forms a devil staircase function (i.e., continuous, monotonic, and the derivatives of E_β with respect to β are large than zero almost everywhere). Moreover, $\lim_{\beta \rightarrow \beta_*} E_\beta = 1$ and $E_2 = 0$.*
- (3) *Let $[\beta_L, \beta_U]$ be the admissible interval generated by a block $t_1 \cdots t_p$ (see [18]). For $\beta \in [\beta_L, \beta_U]$, the escape rate E_β is given by*

$$E_\beta = 1 - \frac{h_{\text{top}}(Z_{t_1 \cdots t_p})}{\log \beta},$$

where $h_{top}(Z_{t_1 \dots t_p})$ is the entropy of the subshift of finite type

$$(3) \quad Z_{t_1 \dots t_p} := \{(d_i) : \overline{t_1 \dots t_p} \leq d_n \dots d_{n+p-1} \leq t_1 \dots t_p, n \geq 1\}$$

and $\bar{\ell} := 1 - \ell$.

Some related results are also addressed herein. In [12], Feng and Sidorov considered the *growth rate* of the points having a continuum of β -expansions, which is somehow a kind of duality of escape rate we study here. Ban *et al.* [4] considered a unimodal map with a symmetric hole in the middle and study the topological entropy of those points whose orbits never fall in the hole under iteration. The authors showed that the entropy function forms a devil's staircase function with respect to the size of the hole. However, the constant part of such function are not completely characterized. Misiurewicz [20] also provided an topological proof for the same result. If $\beta = 2$, Barrera [5] put a symmetry hole about $1/2$ and show that the entropy function forms a devil's staircase with respect to the the size of the hole, and the author completely characterized the constant part of the entropy (also called the *entropy plateau*). The topics of the transitive components of the open systems are discussed in ([5, 6, 7]).

Section 2 is devoted to the proof of Theorem 1.7.

2. PROOF OF THEOREM 1.7

Before proving the main theorem, we provide some useful materials on open systems. Let $f : M \rightarrow M$ be a map which admits a Markov partition \mathcal{P} . We denote by $\pi : \Sigma^{\mathcal{P}} \rightarrow M$ the corresponding coding map and $\Sigma^{\mathcal{P}}$ the corresponding symbolic space with respect to \mathcal{P} . Fix an open hole $H \subset M$, set $\Lambda^* = \Lambda_H^*$ the invariant set of points whose orbits never fall in the hole under f . Denote by $\Sigma^* = \Sigma_H^* = \pi^{-1}\Lambda^*$ the preimage of Λ^* under π and denote by σ the shift map on Σ^* . Let ∂H be the boundary of H . The following result shows that once the boundary points fall in the gap H under iteration of f , then Σ^* is a subshift of finite type.

Proposition 2.1 (Proposition 4.1 [7]). *If for each $x \in \partial H$ there is an i such that $f^i x \in H$, then Σ^* is a SFT.*

- Remark 2.2.** (1) It is worth noting that Δ_β is not open. However, the computation of the Hausdorff dimension and escape rate of \mathcal{U}_q are not affected if we substitute $\Delta_\beta = (1/\beta, 1/\beta(\beta-1))$ since $\overline{\mathcal{U}}_q \setminus \mathcal{U}_q$ is countable [17], where $\overline{\mathcal{U}}_q$ is the closure of \mathcal{U}_q . Thus we define Δ_β as such an open interval in what follows.
- (2) We point out that there is an analogous result of Proposition 2.1 in IFS setting (Theorem 2.4, [3]). The authors in [3] also show that the set of points whose orbits fall in holes is of full Lebesgue measure (Corollary 3.1, [3]).

The following simple proposition reveals that the escape rate are the same under the dynamical systems (J_β, G_β) and (I, T_β) , and the proof is omitted.

Proposition 2.3. *Let $1 < \beta \leq 2$. Then $e_\beta = \tilde{e}_\beta$ and $E_\beta = \tilde{E}_\beta$.*

We denote by Λ_β^* the collection of points whose orbits never fall in the gap Δ_β . Note that Λ_β^* is T_β -invariant. Therefore, $(\Lambda_\beta^*, T_\beta^*)$ is a dynamical system on its own right, where $T_\beta^* = T_\beta|_{\Lambda_\beta^*}$. Also we denote by $\Sigma_\beta^* = \pi_\beta^{-1}\Lambda_\beta^*$ the symbolic space. Let $a = a_\beta := 1/\beta$ and $b = b_\beta := 1/\beta(\beta-1)$. Define $\mathcal{F} := \{\beta \in (1, 2] : a \text{ and } b \text{ fall in } \Delta_\beta \text{ under iteration}\}$ and $\mathcal{N} = I \setminus \mathcal{F}$, i.e., a or b does not fall in the hole Δ_β under T_β for $\beta \in \mathcal{N}$. Proposition 2.1 shows that Σ_β^* is a subshift of finite type for $\beta \in \mathcal{F}$.

2.1. The case where $\beta \in \mathcal{F}$. In this section, we discuss the case where $\beta \in \mathcal{F}$. The aim of this section is to define a new map \mathbf{T}_β which enables us to apply the results of Afraimovich and Bunimovich [1] on open systems. Introduce a piecewise linear map $\mathbf{T}_\beta : I \rightarrow I$ from T_β as follows.

$$\mathbf{T}_\beta(x) = \begin{cases} T_\beta(x), & \text{if } x \notin \Delta_\beta; \\ x, & \text{if } x \in \Delta_\beta. \end{cases}$$

That is, those points which fall in Δ_β under iteration of T_β are **stuck** by \mathbf{T}_β .

For $\beta \in \mathcal{F}$, let $i_* \geq 1$ be such that $T_\beta^{i_*} * \in \Delta_\beta$ and $T_\beta^i * \notin \Delta_\beta$ for $0 \leq i < i_*$, where $*$ stands for a or b . That is, i_* is the first return time

of the orbits of $*$ falls in the hole Δ_β . Denote by $\mathcal{A}_\beta = \{T_\beta^k a\}_{k=0}^{i_a}$ and $\mathcal{B}_\beta = \{T_\beta^k a\}_{k=0}^{i_b}$. Let $\mathcal{C}_\beta = \mathcal{A}_\beta \cup \mathcal{B}_\beta$ be an ordered set in \mathbb{R} . Then \mathcal{B}_β is a partition of I . For $\beta \in \mathcal{F}$, the following lemma provides the explicit representation of the Markov partition of \mathbf{T}_β .

Lemma 2.4. *For $\beta \in \mathcal{F}$, \mathcal{C}_β is a Markov partition for \mathbf{T}_β .*

Proof. It suffices to show that $\mathbf{T}_\beta z \in \mathcal{C}_\beta$ for all $z \in \mathcal{C}_\beta$. We claim that, if $z \in \mathcal{C}_\beta$ and $T_\beta z \notin \Delta_\beta$, then $\mathbf{T}_\beta z = T_\beta z \in \mathcal{C}_\beta$. If this is not the case, then $T_\beta z \in \Delta_\beta$ implies $\mathbf{T}_\beta z = z \in \mathcal{C}_\beta$, which completes the proof. \square

For $\beta \in \mathcal{F}$, let $\eta = \{\eta_j\}_{j \in \mathcal{I}}$ be the Markov partition of \mathbf{T}_β (Lemma 2.4). Decompose the index set $\mathcal{I} = \mathcal{I}_H \cup \mathcal{I}_0$, where $\mathcal{I}_H = \{i \in \mathcal{I} : \eta_i \subseteq \Delta_\beta\}$ and $\mathcal{I}_0 = \mathcal{I} \setminus \mathcal{I}_H$, and let A_β be the corresponding transition matrix, i.e., $A_\beta(i, j) = 1$ if $\mathbf{T}_\beta \eta_i \subseteq \eta_j$ and $A_\beta(i, j) = 0$ otherwise. Denote by A_β^- the 0-1 matrix which is derived by deleting the \mathcal{I}_H -columns and rows of A_β . Let $X_\beta^- = X_{A_\beta^-}$ be the subshift generated by the adjacency matrix A_β^- . Define $\Theta : \mathcal{I} \rightarrow \{0, 1\}$ by $\Theta(i) = 0$ if $\eta_i \subseteq I_0$ and $\Theta(i) = 1$ if $\eta_i \subseteq I_1$. Let $\theta : X_\beta^- \rightarrow \{0, 1\}^\mathbb{N}$ be the map induced by Θ , i.e., $\theta(\omega) = (\Theta(\omega_1), \Theta(\omega_2), \dots)$, where $\omega = (\omega_1, \omega_2, \dots) \in X_\beta^-$. Since the indices of A_β^- are those intervals of $\mathcal{I}_0 = \mathcal{I} \setminus \mathcal{I}_H$, it is evident that

$$(4) \quad h_{\text{top}}(X_\beta^-) = h_{\text{top}}(\Sigma_\beta^*).$$

Let M be a square matrix. We denote by ρ_M the maximal eigenvalue of M . Define $B_\beta = A_\beta \times \text{diag}(1/\beta, \dots, 1/\beta)$, and B_β^- is derived by deleting the \mathcal{I}_H -columns and rows of A_β . The following result in [1] links the escape rate e_β with the entropy $h_{\text{top}}(\Sigma_\beta^*)$.

Lemma 2.5 (Theorem 4, [1]). *The measure $\lambda(\Gamma_{\beta,n})$ satisfies the following asymptotic equality:*

$$\lambda(\Gamma_{\beta,n}) \simeq Q_0(n) \rho_{B_\beta^-}^n \lambda(\Delta_\beta),$$

where $Q_0(n)$ is a polynomial with degree less than the number of the holes m (herein $m = 1$).

Proof of Theorem 1.7 for $\beta \in \mathcal{F}$. Suppose $\beta \in \mathcal{F}$. Since $\rho_{B_\beta^-} = \beta^{-1} \rho_{A_\beta^-}$, Lemma 2.5 is applied to show that

$$\begin{aligned} e_\beta &= \lim_{n \rightarrow \infty} \frac{-\log \lambda(\Gamma_{\beta,n})}{n} = -\log \rho_{B_\beta^-} = \log \beta - \log \rho_{A_\beta^-} \\ &= \log \beta - h_{\text{top}}(X_\beta^-) = \log \beta - h_{\text{top}}(\Sigma_\beta^*). \end{aligned}$$

Since $\dim_H \mathcal{U}_\beta = \frac{h_{\text{top}}(\Sigma_\beta^*)}{\log \beta}$ (Theorem 1.3, [15]), we have

$$E_\beta = \frac{e_\beta}{\log \beta} = 1 - \frac{h_{\text{top}}(\Sigma_\beta^*)}{\log \beta} = 1 - \dim_H \mathcal{U}_\beta.$$

This completes the proof for $\beta \in \mathcal{F}$. \square

2.2. The case where $\beta \in \mathcal{N}$.

Proof of Theorem 1.7. Let $\beta \in \mathcal{N}$, $\Lambda^* = \Lambda_\beta^*$, $T^* = T_\beta^*$, $e = e_\beta$, and $E = E_\beta$. The idea of this proof is to approximate (Λ^*, T^*) by the open systems $\{(\Lambda_l^*, T_l^*)\}_{l=1}^\infty$ such that $\Sigma_l^* = \pi_\beta^{-1} \Lambda_l^*$ is a SFT for all $l \geq 1$.

It suffices to prove the case of $T^n a \notin \Delta_\beta$ for all $n \geq 1$ and b fall in Δ_β under iteration of T . The same argument remains valid for other cases. Since the set of points whose orbits fall in the hole Δ_β is of full Lebesgue measure (Remark 2.2 (ii)), we construct two sequences $\{a_l\}_{l=1}^\infty \subseteq \mathbb{R}$ and $\{i_l\}_{l=1}^\infty \subseteq \mathbb{N}$ such that $a_l \leq a_{l+1}$, $\lim_{l \rightarrow \infty} a_l = a = \frac{1}{\beta}$, $T^{i_l} a_l \in \Delta_\beta$, and $T^i a_l \notin \Delta_\beta$ for $1 \leq i < i_l$. Denote by Λ_l^* the set of points whose orbits never fall in $\Delta_l := (a_l, \frac{1}{\beta(\beta-1)})$. Thus we have $\Lambda_l^* \subseteq \Lambda_{l+1}^*$ and $\Lambda_\beta^* = \overline{\bigcup_{l=1}^\infty \Lambda_l^*}$. Define $\Sigma_l^* = \pi_\beta^{-1} \Lambda_l^*$, Proposition 2.1 infers that Σ_l^* is a SFT (since a_l and $b = \frac{1}{\beta(\beta-1)}$ fall in Δ_l). Construct A_l, A_l^-, B_l, B_l^- , and X_l^- analogously to the case where $\beta \in \mathcal{F}$. Let $\rho_l = \rho_{B_l^-}$. We then have $h_{\text{top}}(\Lambda_l) = h_{\text{top}}(\Sigma_l) = \log \rho_l$ under the same discussion of (4).

Clearly, $\rho_{l+1} \geq \rho_l$. We claim that the sequence $\{a_l\}_{l=1}^\infty$ and $\{i_l\}_{l=1}^\infty$ can be chosen so that $\rho_{l+1} > \rho_l$. Since Σ_l^* is a SFT, Theorem 6.4 in [7] shows that the number of topologically transitive components of $(\Lambda_l^*)^{\text{nw}}$ of Λ_l^* is at most 2 (it is $2r$ actually, where r is the number of holes, and $r = 1$ in our case), where “nw” stands for the non-wandering set of Λ_l^* . Since 0 is a trivial transitive component, thus the number of non-trivial proper topological transitive components with uncountable elements is exact one.

Once the pair (a_l, i_l) has been chosen for $l \geq 1$, we pick a pair (a_{l+1}, i_{l+1}) such that $T^{i_{l+1}}a_{l+1} \in \Delta_{l+1}$ but $T^{i_l}a_{l+1} \notin \Delta_l$. That is, a_{l+1} does not fall in the hole Δ_l for the first i_l iterations. This is possible since $\beta > 1$ and the function $a \rightarrow T^{i_l}a$ grows fast with respect to a if $i_l \geq 1$ is large enough. Therefore, a_{l+1} can be chosen among such those points and wait for its orbit fall in the hole Δ_{l+1} again. For each $l \geq 1$, T_l^* admits a Markov partition and is topologically transitive. Let A_l^- and A_{l+1}^- be the corresponding adjacency matrices. We may assume that A_l^- and A_{l+1}^- are of the same size, otherwise one can present both of them by N th higher block representation (Definition 1.4.1, [19]) until the lengths of all forbidden sets in X_l^- and X_{l+1}^- are less than N for some $N \in \mathbb{N}$. From the construction of (a_{l+1}, i_{l+1}) above we know that there exists at least a word which belongs to X_{l+1}^- but not belong to X_l^- . That is, $A_l^- < A_{l+1}^-$. Thus, we have $B_l^- < B_{l+1}^-$. Theorem 4.4.7 in [19] indicates that $\rho_l < \rho_{l+1}$.

Since $\Lambda^* = \overline{\bigcup_{l=1}^{\infty} \Lambda_l^*}$, it follows from Lemma 4.1.10 in [2] that

$$h_{top}(\Lambda^*) = \sup_l h_{top}(\Lambda_l^*) = \lim_{l \rightarrow \infty} h_{top}(\Lambda_l^*).$$

Let e_l (resp. E_l) be the escape rate corresponding to the hole Δ_l . From Lemma 2.5, we deduce that, for all $l \geq 1$,

$$e_l = \log \beta - \log \rho_l \text{ and } E_l = \frac{e_l}{\log \beta} = 1 - \frac{\log \rho_l}{\log \beta} = 1 - \frac{h_{top}(\Lambda_l^*)}{\log \beta}.$$

Since e_l is continuous on l , so is E_l . Taking $l \rightarrow \infty$ we then have

$$E = \lim_{l \rightarrow \infty} E_l = 1 - \frac{1}{\log \beta} \lim_{l \rightarrow \infty} h_{top}(\Lambda_l^*) = 1 - \frac{1}{\log \beta} h_{top}(\Lambda^*) = 1 - \dim_H \mathcal{U}_\beta.$$

The last equality comes from the fact that $\dim_H \mathcal{U}_\beta = \frac{h_{top}(\Lambda^*)}{\log \beta}$ for general subshift (Theorem 1.3, [15]). That is, $\dim_H \mathcal{U}_\beta + E = 1$, which establishes the formula. \square

Proof of Corollary 1.8. (1) and (3) of Corollary 1.8 are the immediate consequences of the following facts: (i) $\dim_H \mathcal{U}_\beta = 0$ for $1 < \beta \leq \beta_*$, (ii) $\dim_H \mathcal{U}_2 = 1$, and (iii) $\dim_H \mathcal{U}_\beta = \frac{h_{top}(Z_{t_1 \dots t_p})}{\log \beta}$ for $\beta \in [\beta_L, \beta_U]$, where $h_{top}(Z_{t_1 \dots t_p})$ is the topological entropy of the SFT (3) (Theorem 2.6, [18]). Finally, combining Theorem 1.7 with the fact that $q \rightarrow \dim_H \mathcal{U}_\beta$

forms a devil staircase function (Theorem 1.7, [15]) yields (2). This completes the proof. \square

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